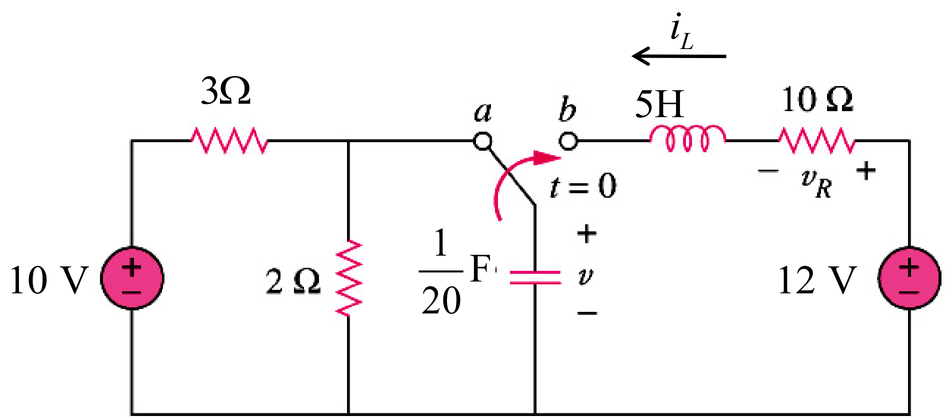
These problems are typical of, but harder than, exam problems.

1: Find $v(t), t > 0$



Initial/final conditions:

After the switch is thrown, this is a 2nd order series R-L-C circuit.

$v(t) = v_T(t) + v_{SS}(t)$

To find $v_{SS}(t)$, we look a long time after the switch is thrown. The capacitor looks like an open circuit, so no current flows, and the voltage is

$\boxed{v_{SS}(t) = v(\infty) = 12\text{V}.}$

To find $v_T(t)$, assume that the 12 V is turned off. Then we have a source free R-L-C circuit.

The initial conditions

$$v(0^-) = 10\text{V} \times \frac{2\Omega}{2\Omega + 3\Omega} = 4\text{V}. \text{ (voltage divider, capacitor is open circuit)}$$

The voltage across a capacitor may not change abruptly, so $\boxed{v(0^+) = v(0^-) = 4\text{V}.}$

The inductor current $i_L(0^-)$ is zero because with the switch in position B, no current flows in the right hand side of the circuit. The current through an inductor can't change abruptly, so $i_L(0^+) = 0$ A. With the switch in position B, the elements are all connected in series, so $i_C(0^+) = 0 \text{ A} = C\frac{dv}{dt}$, so $\boxed{\frac{dv}{dt} = 0\text{V}}$.

For the currents, we already have $\boxed{i_L(0^+) = 0 \text{ A}.}$

$\boxed{i_L(\infty) = 0,}$ as the capacitor becomes an open circuit.

From KVL at $t = 0^+, -v_C - v_L - v_R + 12 = 0$ where all values are at 0^+. $v_C(0^+) = 4\text{V}$, and $v_R(0^+) = i_L(0^+)R = 0$. So KVL gives $-4 - v_L - 0 + 12 = 0$, or $v_L(0^+) = 8\text{V}$.

$$v_L = L\frac{di}{dt}, \text{ so } \frac{v_L}{L} = \boxed{\left.\frac{di}{dt}\right|_{0^+} = \frac{8 \text{ V}}{5\text{H}} = 1.6 \text{ A/sec}}$$

For solution, this is a Series LRC circuit

$$\alpha = \frac{R}{2L} = \frac{10\Omega}{2\times 5\text{H}} = 1/\text{s}$$

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{5\text{H}\times 0.05\text{F}}} = \sqrt{\frac{1}{0.25\ \text{s}^2}} = 2/\text{s}$$

This is underdamped $\omega_0 > \alpha, \omega_D = \sqrt{2^2 - 1^2} = \sqrt{4-1} = 1.73$

$$v(t) = v_{ss} + e^{-\alpha t}\left(A\cos\omega_D t + B\sin\omega_D t\right) = 12\text{V} + e^{-t}\left(A\cos 1.73t + B\sin 1.73t\right)$$

$$v(0) = 4\text{V} = 12 + 1\times\left(A\times 1 + B\times 0\right) = 12 + A \Rightarrow A = -8\text{V}$$

$$\left.\frac{dv}{dt}\right|_{t=0^+} = 0 = \left[-e^{-t}\left(A\cos 1.73t + B\sin 1.73t\right) + e^{-t}\left(-1.73A\sin 1.73t + 1.73B\cos 1.73t\right)\right]_{t=0}$$

$$0 = -1\left(A + B\times 0\right) + 1\left(-1.73A\times 0 + 1.73B\times 1\right) = -A + 1.73B = 8 + 1.73B$$

$$B = -8/1.73 = -4.62$$

$$\boxed{v(t) = 12 + e^{-t}\left(-8\cos 1.73t - 4.62\sin 1.73t\right)}$$

$$v(0) = 12 + (-8 - 0) = 4\text{V}$$

$$v(\infty) = 12 + 0 = 12\text{V}$$

$$\left.\frac{dv}{dt}\right|_{t=0^+} = \left[-1\left(-8+0\right) + 1\left(0 + 1.73(-4.62)\right)\right] = 8 - 1.73(4.62) = 0$$

2:

In the previous circuit, find $i_L(t), t > 0$.

The L and C are in series so $i_L(t) = i_C(t) = C\frac{dv_C}{dt}$. Using the solution just completed

$$\boxed{v(t) = 12 - e^{-t}\left(8\cos 1.732t + 4.6\sin 1.732t\right)\text{V}}$$

$$\frac{dv}{dt} = (-1)(-1)e^{-t}\left(8\cos 1.732t + 4.6\sin 1.732t\right)$$

$$+(-1)e^{-t}\left(-8(1.732)\sin 1.732t + 4.6(1.732)\cos 1.732t\right)$$

$$= e^{-t}\left(8 - 4.6(1.732)\right)\cos 1.732t + e^{-t}\left(4.6 + 8(1.732)\right)\sin 1.732t$$

$$= e^{-t}\left(0\cos 1.732t + 18.48\sin 1.732t\right) = 18.48e^{-t}\sin 1.732t.$$

$$\boxed{i_L(t) = 0.05\text{s}/\Omega \times 18.48e^{-t}\sin 1.732t = 0.9238e^{-t}\sin 1.732t}$$

Quick check of conditions:

$i(0) = 0$

$i(\infty) = 0$

$$\frac{di}{dt}@0 = -0.9238e^{-0}\sin 1.732\times 0 + 0.9238e^{-0}(1.732)\cos 1.723\times 0$$

$= 1.6\text{A/s}.$

Things check!

Alternatively, the solution for $i_L(t)$ must have the same form as the solution for $v_C(t)$

$$i_L(t) = i_L(\infty) + e^{-\alpha t}\left(A\cos\omega_d t + B\sin\omega_d t\right)$$

$$= 0 + e^{-t}\left(A\cos 1.732t + B\sin 1.732t\right)$$

$i_L(0) = 0 = 0 + 1\left(A\times 1 + B\times 0\right)$, so $A = 0$.

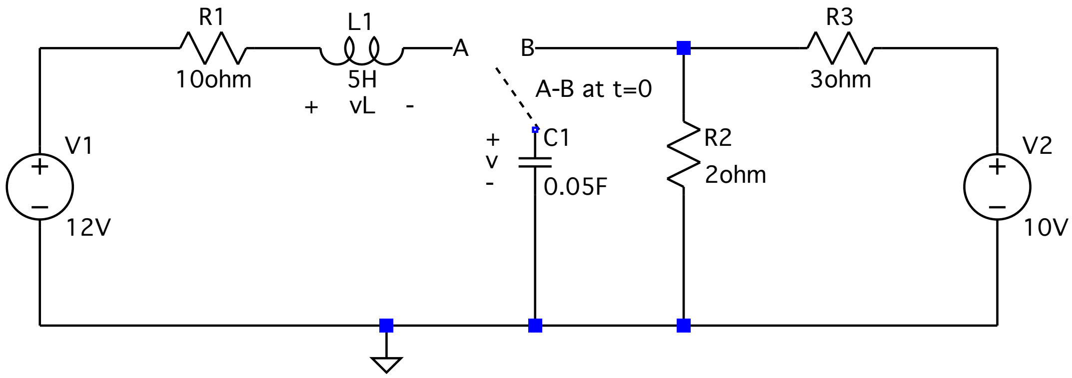
$$\left.\frac{di_L}{dt}\right|_{t=0} = 1.6\text{A/s} = -\alpha A + \omega_d B = 0 + 1.732B$$

$$B = \frac{1.6\text{A/s}}{1.732/\text{s}} = 0.9238\text{A}$$

So the solution is

$\boxed{i_L(t) = 0.9238e^{-t}\sin 1.732t}$, as before!

3: The switch (dotted line) moves **from A to B** at $t = 0$. Determine $v(t),\ t > 0$



With the switch in position B, it is a
first order circuit, so we need $v(0^+), v(\infty)$, and τ.

With the switch in position A, the capacitor is an open circuit, no current flows,
so there is no voltage drop. $v(0^-) = 12\text{V} = \text{v}(0^+)$

With the switch in position B, at $t = \infty$ the capacitor is an open circuit,
3Ω in series with 2Ω forms a voltage divider, with $v =$ voltage across the 2Ω

$$v(\infty) = 10\text{V} \times \frac{2\Omega}{3\Omega + 2\Omega} = 4\text{V}.$$

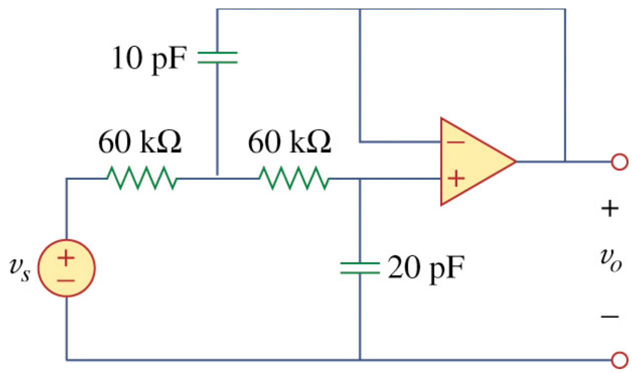
Turn the voltage source off and we see that the capacitor "sees" $2\Omega \parallel 3\Omega = \frac{6}{5}\Omega$

$$\tau = RC = 1.2\Omega \times \frac{1}{20}\text{F} = 0.06\text{s}.$$

$$v(t) = 4 + (12 - 4)e^{-t/0.06} = 4 + 8e^{-t/0.06}$$

4: **One of my favorite problems. Work carefully and it works! Also, it gives practice constructing the differential equations. (This one will NOT be on the exam, but work on it anyway!)**

Obtain the differential equation for the op-amp circuit shown below. This is a second-order op-amp circuit. *Hint: Solve the problem in terms of generic variables* R_1, R_2, C_1, C_2, *then substitute the given values. Remember the differential equation for the capacitors! Remember your op-amp rules!*


10 pF
60 kΩ
60 kΩ
−
+
v_s
+
−
20 pF
+
v_o
−

Define node 1 on the left and node 2 on the right, closer to the op-amp input.
Define $R = 60\text{k}\Omega$. Let C_1=10pF and C_2=20pF. At node 1, KCL:

$$\frac{v_S - v_1}{R} = \frac{v_1 - v_2}{R} + i_{C1},\ i_{C1} = C_1\frac{dv_{C1}}{dt} = C_1\frac{d}{dt}(v_1 - v_2)$$

or $\dfrac{v_S - v_1}{R} = \dfrac{v_1 - v_2}{R} + C_1\dfrac{dv_1}{dt} - C_1\dfrac{dv_2}{dt}$

But $v_2 = v_0$ due to the feedback path and the ideal op-amp assumption.
Substituting and multiplying by R

$$v_s - v_1 = v_1 - v_0 + RC_1\frac{dv_1}{dt} - RC_1\frac{dv_0}{dt}$$

$$v_s = 2v_1 - v_0 + RC_1\frac{dv_1}{dt} - RC_1\frac{dv_0}{dt} \text{ (Equation 1)}$$

At node 2, KCL

$\dfrac{v_1 - v_2}{R} = i_- + i_{C2} = 0 + C_2\dfrac{dv_2}{dt}$. Multiplying by R and substituting $v_2 = v_0$

$$v_1 = v_0 + RC_2\frac{dv_0}{dt} \text{ (Equation 2).}$$

Finally, substituting Equation 2 into Equation 1

$$v_s = 2\left(v_0 + RC_2\frac{dv_0}{dt}\right) - v_0 + RC_1\frac{d}{dt}\left(v_0 + RC_2\frac{dv_0}{dt}\right) - RC_1\frac{dv_0}{dt}$$

$$v_S = v_0 + 2RC_2\frac{dv_0}{dt} + RC_1RC_2\frac{d^2v_0}{dt^2} = v_0 + 2RC_2\frac{dv_0}{dt} + R^2C_1C_2\frac{d^2v_0}{dt^2}$$

You could leave it this way and be correct, but our standard form has the leading coefficient of $\dfrac{d^2v_0}{dt^2}$ as 1, so we should divide all of the terms by $R^2C_1C_2$ to get:

$$\frac{d^2v_0}{dt^2} + \left(\frac{1}{RC_1}\right)\frac{dv_0}{dt} + \frac{2v_0}{R^2C_1C_2} = \frac{v_S}{R^2C_1C_2}$$

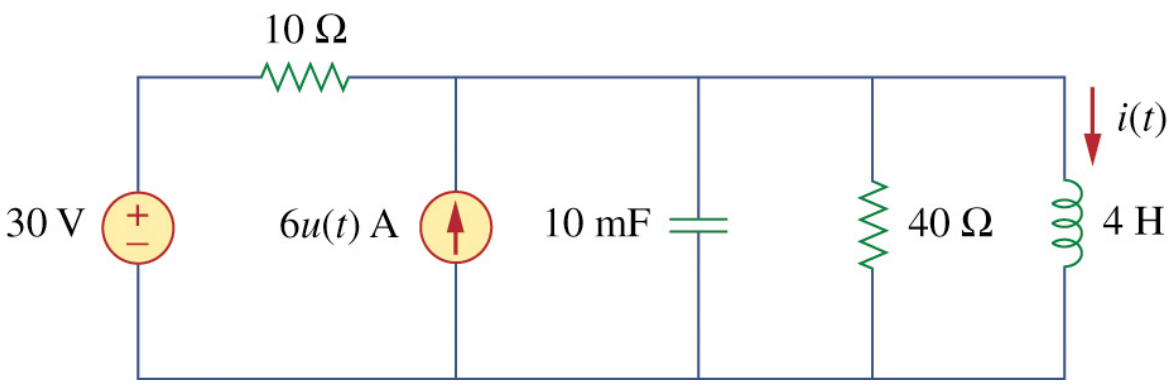
The units work out as Volts/sec^2, as you can (and should) verify.
Substituting $R = 60\text{k}\Omega, C_1 = 10\text{pF}, C_2 = 20\text{pF}$

$RC_1 = 6\times10^{-4}$ sec,$R^2C_1C_2 = 7.2\times10^{-7}$ sec^2

$$\frac{d^2v_0}{dt^2} + \left(\frac{2}{6\times10^{-4}\text{ sec,}}\right)\frac{dv_0}{dt} + \frac{v_0}{7.2\times10^{-7}\text{ sec}^2} = \frac{v_S}{7.2\times10^{-7}\text{ sec}^2}$$

$$\boxed{\frac{d^2v_0}{dt^2} + (3333)\frac{dv_0}{dt} + 1.389\times10^6 v_0 = 1.389\times10^6 v_S}$$

5: Find $i(t)$, $t>0$. The expression $u(t)$ on the 6A source means that the source is OFF for $t<0$ and turns ON with a value of 6A at $t=0$. *Hint: Remember what it means to turn off a current source.*



When the current source is off (for $t<0$), the branch is an open circuit.
For $t<0$, the capacitor is open, the inductor is a short. The voltage across a short circuit is zero, and the inductor, resistor and capacitor are in parallel.
Therefore, $v_C(0^-) = 0\text{V}$ (we'll need this in a minute).
With the 6A source off, the capacitor open, and the inductor short, we have

$$i(0^-) = \frac{30\text{V}}{10\Omega} = 3\text{A} = i(0^+).$$

At $t=\infty$, the current source is on, but the capacitor is open and the inductor is a short. By KCL, $i(\infty) = \dfrac{30\text{V}}{10\Omega} + 6\text{A} = 9\text{A}.$

We use KVL to find v_L, and thence $\dfrac{di}{dt} = \dfrac{v_L}{L}$. The voltage across the capacitor can't change abruptly, and the inductor and capacitor are in parallel, so

$$v_L(0^+) = v_C(0^+) = 0.\text{Therefore } \left.\frac{di}{dt}\right|_{t=0^+} = 0\text{A/s}.$$

For $t > 0$, use Thevenin to determine the resistance seen by the L and R.

Turning the 30V (becomes a wire) and the 6A(open), we have $C \parallel L \parallel \left(40\Omega \parallel 10\Omega\right)$

$40\Omega \parallel 10\Omega = \frac{400}{50} = 8\Omega.$

For a parallel RLC, $\alpha = \frac{1}{2RC} = \frac{1}{2\times 8\Omega \times 10\text{mF}} = \frac{1}{16\times 0.01} = 6.25\text{s}$

$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10\times 10^{-3}\text{F}\times 4\text{H}}} = 5.\ s_{12} = -6.25 \pm \sqrt{6.25^2 - 5^2} = -6.25 \pm 3.75$

$= -10, -2.5$

$\alpha > \omega_0$ implies an overdamped solution

$i(t) = Ae^{s_1 t} + Be^{s_2 t} + i(\infty) = Ae^{-10t} + Be^{-2.5t} + 9\text{A}$

$i(0) = A + B + 9 = 3\text{A}$ <=== corrected

$\left.\frac{di}{dt}\right|_{t=0} = 0 = \left. As_1 e^{s_1 t} + Bs_2 e^{s_2 t}\right|_{t=0} = A(-10) + B(-2.5)$

$\begin{bmatrix} 1 & 1 \\ -10 & -2.5 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \end{bmatrix}$

$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 2 \\ -8 \end{bmatrix}$

$i(t) = 2e^{-10t} - 8e^{-2.5t} + 9, t > 0$

Check at $t = 0: 2 - 8 + 9 = 3$ OK

Check at $t = \infty$: $0 - 0 + 9 = 9$ OK

Check derivative: $2(-10) - 8(-2.5) = -20 + 20 = 0$ OK.

Solution of matrix equation with MATLAB

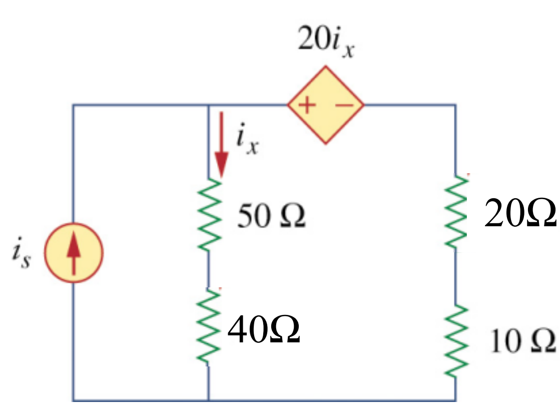
```
>> A=[1 1;-10 -2.5]

A =

   1.0000    1.0000

 -10.0000   -2.5000

>> b=[-6;0]

b =

   -6

    0
```

```
>> ABcoefficients = A\b

ABcoefficients =

     2

    -8
```

6:

For the circuit shown below, assume that $i_s(t) = 6\cos 10^3 t$ A.

I want you to do this problem two ways. First, replace the capacitor with a 40Ω resistor, and the inductor with a 20Ω resistor. Compute the current i_x and the voltage generated by dependent source. This should be pretty easy, as we covered it all before the first exam. *Hint: What is the effect of a resistor on a sinusoidal current or voltage?*



First, replace $40\mu F$ with 40Ω, and 20mH with 20Ω.

Solve with meshes to avoid the supernode.

$i_1 = i_S$ because the source is not shared with any other mesh.

$i_X = i_1 - i_2$ because i_X is shown in the i_1 direction, but both currents flow through the $50\Omega + 40\Omega$ branch.

$$(40\Omega + 50\Omega)(i_2 - i_1) + 20i_X + (20\Omega + 10\Omega)i_2 = 0$$

Substitute for i_X and $i_1$and combine common terms

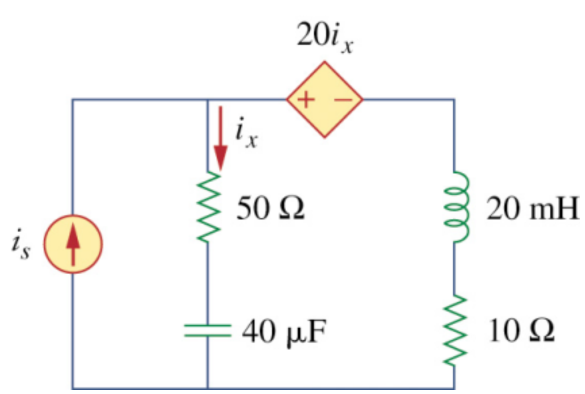
$$(40\Omega + 50\Omega - 20)(i_2 - i_S) + (20\Omega + 10\Omega)i_2 = 0$$

$$(40 + 50 - 20 + 20 + 10)i_2 = (40 + 50 - 20)i_S$$

$i_2 = \frac{70\Omega}{100\Omega} i_S$. Resistors do not change the phase of the input, because they have no reactive parts. $i_2 = 0.7 \times 6\cos(10^3 t) = 4.2\cos 10^3 t$

$i_X = i_1 - i_2 = 1.8\cos 10^3 t$.

Second, using the original circuit, use the angular frequency of the current source to convert $40\mu F$ and 20mH to their complex impedances. (*Hint: both will be strictly imaginary, with no real part. We call the imaginary part of the impedance the reactance.*) Treat these reactances in the same way as resistances (just imaginary) and repeat the problem. The process is the same, but the computations now require complex arithmetic.



Part 2 is exactly the same, but with the complex impedances for the capacitor and inductor.

Then (sorry, I didn't use vector notation, but all these things are complex!)

KVL in left mesh is unnecessary, as $i_1 = i_s$.

KVL in right mesh is

$Z_1(i_2 - i_1) + 20i_x + Z_2 i_2 = 0$. But $i_x = i_1 - i_2 = i_s - i_2$, so $i_2 = i_S - i_x$

$-Z_1 i_x + 20i_x + Z_2(i_s - i_x) = 0$

$(Z_1 + Z_2 - 20)i_x = Z_2 i_s$

$i_x = \dfrac{Z_2 i_s}{(Z_1 + Z_2 - 20)}$. Compare this to part 1 to see it's exactly the same equation!

$$Z_1 = 50\Omega - j\frac{1}{10^3 \text{ r/s} \times 40 \times 10^{-6} \text{mho-s}} = 50\Omega - j\frac{1}{10^3 \text{ r/s} \times 40 \times 10^{-6} \text{mho-s}}$$

$$= (50 - j25)\Omega = 25(2 - j)\Omega$$

$$Z_2 = 10\Omega + j10^3 \text{ r/s} \times 20 \times 10^{-3} \Omega\text{-s} = 10(1 + j2)\Omega$$

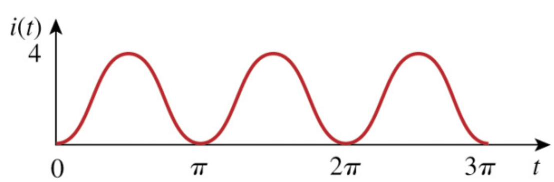
$$i_x = \frac{(10 + j20)\Omega}{(50 - j25)\Omega + (10 + j20)\Omega - 20} \times 6\text{A}\angle 0$$

$$= 1.1 + j3.14\,\text{A} = 3.32\text{A}\angle 70.56$$

$$20\mathbf{i}_x = 20 \times 3.32\angle 70.56 = 66.4\angle 70.56^\circ \text{ V}$$

7:

Write the rectangular (real plus imaginary), phasor, exponential and time domain forms of the following waveform. You can assume that the waveform is sinusoidal in form. There are multiple possible answers, you only need provide one.



There is clearly a sinusoid plus a bias (offset) DC current.

$$i_{BIAS} = \frac{\max(i(t)) + \min(i(t))}{2} = \frac{4+0}{2} = 2\text{A}$$

The amplitude of the sinusoid is $I_m = \frac{i_{PEAKtoPeak}}{2} = \frac{\max(i(t)) - \min(i(t))}{2} = 2\text{A}$

By convention, we want a $\cos(\omega t \pm \phi)$ form. I recommend that you convert all input currents and voltages to this form before proceding.

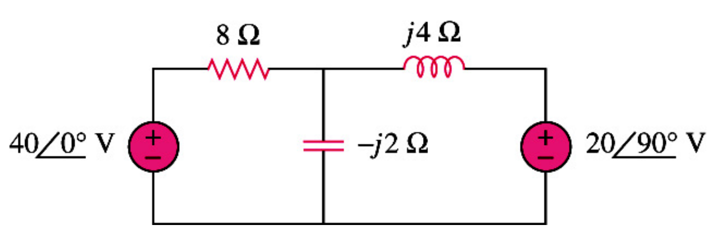
From the plot, the period is $T = \pi$ seconds. $\omega = \frac{2\pi}{T} = \frac{2\pi}{\pi} = 2$ rad/sec

From the plot, we see that the cosine wave starts at $t = \frac{\pi}{2}$ seconds.

At the angular frequency ω, the phase corresponding to a time interval, τ, is $\phi = \omega\tau$. In this case, $\phi = \omega\frac{\pi}{2} = 2\times\frac{\pi}{2} = \pi$ radians. The cosine wave is shifted *right*, so $i(t) = 2 + 2\cos(2t - \pi) = 2 - 2\cos(2t) = 2 + 2\angle\pi$. We can't simplify it any further.

8:

Find the voltage across the capacitor. The voltages are sinusoidal, because they are written as phasors. We always take a 0° phase to be a cosine. The frequency doesn't matter, because the $\mathbf{Z}_L = j\omega L$ and $\mathbf{Z}_C = \frac{1}{j\omega C}$ conversions have already been done.



We don't need to convert to impedances, as that has already been done.

Using meshes

Left hand mesh:

$$-40\angle 0 + \mathbf{i}_1 \times 8\Omega + (\mathbf{i}_1 - \mathbf{i}_2)(-j2\Omega) = 0$$

$$\left(8 - j2\right)\mathbf{i}_1 - (-j2)\mathbf{i}_2 = 40 + j0$$

$$\left(8 - j2\right)\mathbf{i}_1 + (j2)\mathbf{i}_2 = 40 + j0$$

Right hand mesh

$$\mathbf{i}_2 \times (j4) + 20\angle 90^\circ + (\mathbf{i}_2 - \mathbf{i}_1)(-j2) = 0$$

$$-(-j2)\mathbf{i}_1 + (j4 - j2)\mathbf{i}_2 = -20\angle 90^\circ = 20\angle -90^\circ = -j20$$

$$(j2)\mathbf{i}_1 + (j2)\mathbf{i}_2 = -20\angle 90^\circ = 20\angle -90^\circ = -j20$$

$$\begin{bmatrix} 8 - j2 & j2 \\ j2 & -j2 \end{bmatrix} \begin{bmatrix} \mathbf{i}_1 \\ \mathbf{i}_2 \end{bmatrix} = \begin{bmatrix} 40 \\ -j20 \end{bmatrix}$$

(note this is the same matrix equation we get with inspection using **Z**, not *R*. Try it!)

MATLAB

$$\begin{bmatrix} \mathbf{i}_1 \\ \mathbf{i}_2 \end{bmatrix} = \begin{bmatrix} 5.0000 - j2.5000\text{i} \\ 15.0000 - j2.5000\text{i} \end{bmatrix} = \begin{bmatrix} 5.5902\angle -26.6^\circ \\ 15.2069\angle -9.4^\circ \end{bmatrix}$$

The voltage across the capacitor is $(\mathbf{i}_1 - \mathbf{i}_2) \times (-j2) = j20\text{V} = 20\angle 90^\circ\,\text{V}$