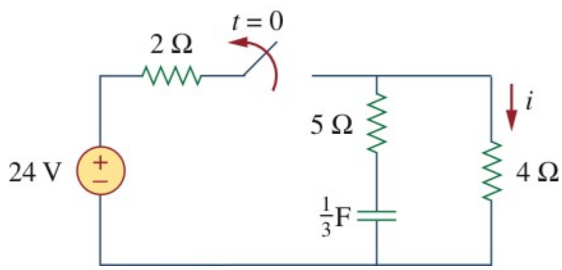
1: First order review

For the following circuit, find $i(t),\ t > 0.$



All voltages and currents in a circuit have the same mathematical form, but not the same values. At $t = 0^-$, the switch is closed. No current flows through 5Ω, so there is no voltage drop. The voltage across the capacitor is therefore the voltage across the 4Ω, $v_C = 24\text{V} \times \dfrac{4\Omega}{4\Omega + 2\Omega} = 16\text{V}.$

With the switch open, the circuit is source-free, so $v_C(\infty) = 0.$

With the switch open, $\tau = RC = 9\Omega \times \left(\dfrac{1}{3}\text{F}\right) = 3\text{s}.$

$v_C(t) = v(\infty) + \left(v(0) - v(\infty)\right)e^{-t/\tau} = 0 + \left(16 - 0\right)e^{-t/3} = 16e^{-t/3}\text{V}.$

From Ohm's Law, $i(t) = \dfrac{v_C(t)}{9\Omega} = \dfrac{16}{3}e^{-t/3}\text{A},\ t > 0.$

2: First order review

In the previous circuit, assume that the switch is operated in the opposite direction and *closes* at $t = 0$. Find $i(t), t > 0$.

This problem requires some careful thought. The *current* cannot change abruptly in an inductor. The *voltage* cannot change abruptly in a capacitor. But both current and voltage can change abruptly in a resistor!

There is one capacitor, so this is a first order circuit. All currents in the circuit, and all voltages except the source, obey the same form of LCCDE, and, therefore, have the same form of solution.

$i(t)$ is a resistor current. Therefore, there is no relationship between $i(0^-)$ and $i(0^+)$. For our initial condition, we need to find $i(0^+)$. When the switch is open $(t < 0)$, the capacitor is part of a source free circuit, so $v_C(0^-) = 0\text{V}$.
When the switch closes, 24V is applied to the circuit. Both the 5Ω and 4Ω resistors are connected to 0V because $v_C(0^+) = v_C(0^-)$. Thus the source sees $2\Omega + (4 \| 5) = 2 + \frac{20}{9} = \frac{38}{9} = 4\frac{2}{9} = 4.22\Omega$. The current is $i_{2\Omega} = \frac{24\text{V}}{(38/9)\Omega} = \frac{108}{19}\text{A}$.
This current is divided between the 5Ω path, which includes the capacitor, and the 4Ω path. $i(t)$ is the 4Ω path, so $i(0^+) = i_{2\Omega} \times \frac{5}{4+5} = \frac{108}{19}\text{A} \times \frac{5}{9} = \frac{60}{19}\text{A}$.

At $t=\infty$, the capacitor is open and no current flows across the 5Ω path.

Thus $i(\infty)=\dfrac{24\text{V}}{2\Omega+4\Omega}=4\text{A}$.

When the switch is closed, we compute the Thevinin equivalent resistance seen by the capacitor. Turn the 24V source off and we have

$$5\Omega+\left(2\,\|\,4\right)=5\Omega+\frac{8}{6}\Omega=5\Omega+\frac{4}{3}\Omega=\frac{19}{3}\Omega.$$

$$\tau=RC=\frac{19}{3}\Omega\times\frac{1}{3}\frac{\text{s}}{\Omega}=\frac{19}{9}\text{s}$$

Plugging into our standard 1st order solution:

$$i(t)=i(\infty)+\left(i(0)-i(\infty)\right)e^{-t/\tau}=4+\left(\frac{60}{19}-4\right)e^{-9t/19}$$

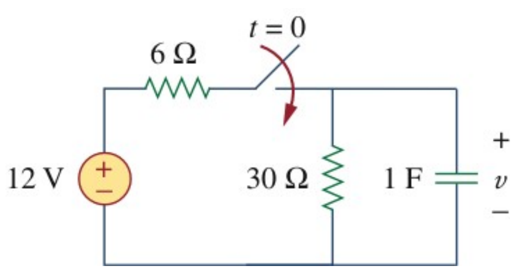
$$=4-0.84e^{-9t/19}\text{A},\ t>0.$$

Let's check the conditions. $t=0, i(0)=4-(0.84\times 1)=3.16=\dfrac{60}{19}\text{A}=i(0^{+})$

$t=\infty, i(\infty)=4-(0.84\times 0)=4\text{A}=i(\infty)$.

3: First order review

In the following circuit, find $v(t), t>0$. Then find the current, $i(t)$, across the 30Ω resistor for $t>0$. What do you notice about the form of $v(t)$ and $i(t)$?



the switch is open at $t=0$, so $v(0^-)=0$.

At $t=\infty$, the capacitor is open, so we have a voltage divider.

$$v_C(\infty)=12\text{V}\times\frac{30\Omega}{36\Omega}=10\text{V}.$$

After the switch is closed, use Thevenin to find the equivalent resistance.

$$30\Omega \,\|\, 6\Omega=\frac{180\Omega^2}{36\Omega}=5\Omega,\ \tau=RC=5\Omega\times 1\text{F}=5\text{s}$$

$$v(t)=v(\infty)+\big(v(0)-v(\infty)\big)e^{-t/\tau}=10+\big(0-10\text{V}\big)e^{-t/5\text{s}}=10\big(1-e^{-t/5\text{s}}\big)\text{V},\ t>0$$

30Ω is in parallel with the capacitor, so $v_{30\Omega}=v_C=v(t)$.

$$i_{30\Omega}=\frac{v_C(t)}{30\Omega}=\frac{10\big(1-e^{-t/5\text{s}}\big)\text{V}}{30\Omega}=\frac{1}{3}\big(1-e^{-t/5\text{s}}\big)\text{A},t>0.$$

The forms of the expressions are the same, but the constants are different.

4: First order review

Repeat the previous problem, assuming that the switch has been closed a long time and opens at $t=0$.

Again, the initial and final conditions are swapped. $v(0)=10\text{V}$, $v(\infty)=0$.

With the switch open, the circuit is source free (thus $v(\infty)=0$), and with only 30Ω. $\tau=RC=30\Omega\times 1\text{F}=30\text{s}$.

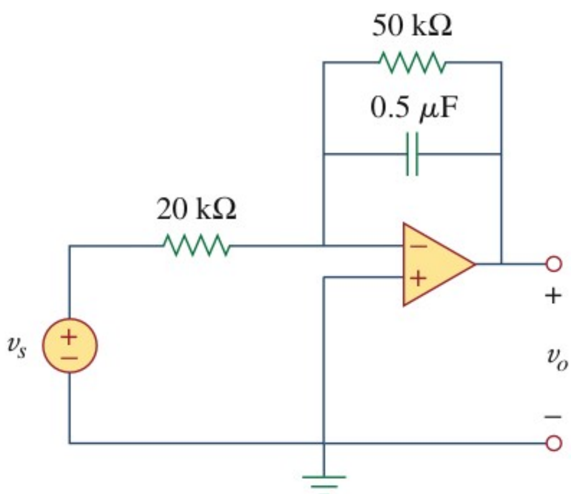
$$v(t)=0+(10-0)e^{-t/30}\text{V},t>0$$

$$i(t)=\frac{v(t)}{30\Omega}=\frac{1}{3}e^{-t/30}\text{A},t>0$$

5: First order op amp DO NOT PANIC!

For the op-amp circuit shown here, assume that
$v_s(t) = 0$ for $t < 0$ and suddenly jumps to 1V at $t = 0$. Compute $v_0(t)$.

Hint: For a capacitor circuit, the initial condition is the voltage across the capacitor at $t = 0^-$.
Hint: Do Not Panic. Remember the 2 facts you know about an ideal op-amp. Then work from there.



$v_+ = v_- = 0.$

At $t = 0, v_S = 0$ (source is off), and the capacitor is open. No current flows over 20kΩ ($v_S = 0$), so no current flows across 50kΩ, so there is no voltage drop, so $v_0(0^-) = 0.$

At $t = \infty$, the capacitor is open, and we have a standard inverting op amp with gain $\frac{v_0}{v_S} = \frac{50\text{k}\Omega}{20\text{k}\Omega} = 2.5.$ So $v_0(\infty) = 2.5\text{V}.$

For $\tau = RC$, the voltage source is off, and the capacitor only sees 50kΩ.

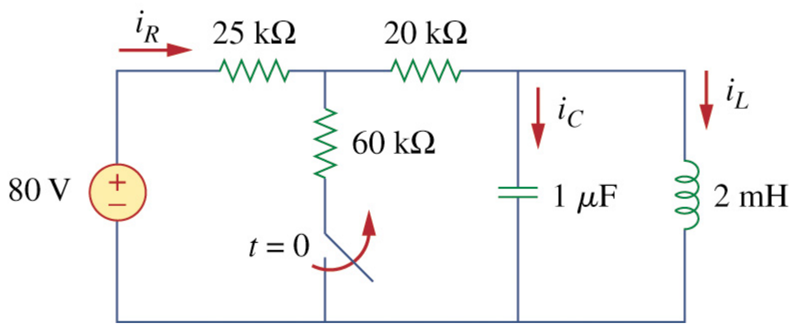
$\tau = RC = 50\times10^3\Omega\times0.5\times10^{-6}\text{F}=25\times10^{-3}\text{s} = 25\text{ms}$

$v_0(t) = v(\infty) + \big(v(0) - v(\infty)\big)e^{-t/\tau}$

$= 2.5 + (0 - 2.5)e^{-t/0.025} = 2.5\left(1 - e^{-40t}\right), t > 0$

7: Second order initial conditions

Find $i_R(0^+), i_L(0^+), i_C(0^+), di_R(0^+)/dt, di_L(0^+)/dt, di_C(0^+)/dt, i_R(\infty), i_L(\infty), i_C(\infty)$ (9 initial and final conditions). This problem is the equivalent of three initial condition problems. *Hint: The resistor has no associated differential equation, but all of the currents and voltages have the same form. Use KCL & KVL as appropriate. Take your time and work through it.*



This is one of the hardest homework problems I assign, and is much harder than any exam questions!

$t = 0^+$ Solutions

For $t = 0^-$, the capacitor is an open circuit and the inductor is a short circuit.

No current flows over the capacitor.

Therefore, the 60k and 20k resistors are in parallel, forming a current divider.

The equivalent resistance of $60\text{k}||20\text{k} = \dfrac{60\text{k} \times 20\text{k}}{60\text{k}+20\text{k}} = \dfrac{1200\text{k}^2}{80} = 15\text{k}\Omega.$

The series combination of 25k and 15k is 40kΩ. Therefore, $i_R(0^-) = \dfrac{80\text{V}}{40\text{k}\Omega} = 2$ mA.

This current is divided by the parallel resistors. $i_L(0^-)$ is equal to the current flowing through the 20kΩ resistor, $i_L(0^-) = 2\text{mA} \times \dfrac{60\text{k}}{60\text{k}+20\text{k}} = 1.5$ mA. The current through the inductor cannot change abruptly, therefore $\boxed{i_L(0^+) = 1.5\text{mA}.}$

The capacitor is an open circuit, while the inductor is a short circuit. Therefore $v_C(0^-) = 0\text{V}$. (But we didn't have to find this! But you will need it, stay tuned!)

Now, when the switch initially opens, the 60k resistor is removed from the circuit. The voltage across the capacitor $v_C(0^+) = v_C(0^-) = 0\text{V}$. Without the 60k resistor, we have 25k+20k=45k, and $\boxed{i_R(0^+) = \dfrac{80\text{V}}{45\text{k}\Omega} = 1.78\text{mA}}$.

We've already established $i_L(0^+) = 1.5\text{mA}$, so KCL gives $\boxed{i_C(0^+) = 1.78 - 1.5 = 0.28\text{mA}}$

$t = \infty$ solutions

The capacitor is open and the inductor is shorted.

$$\boxed{i_R(\infty) = \frac{80\text{V}}{45\text{k}\Omega} = 1.78\text{mA}}$$

The capacitor is an open circuit, $\boxed{i_C(\infty) = 0\text{mA}}$

Because the capacitor is open, $\boxed{i_L(\infty) = i_R(\infty) = 1.78\text{mA}}$

The derivative solutions at $t = 0^+$

The voltage across the capacitor can't change abruptly. From the $t = 0^+$ solution, $+v_C(0^+) = 0$. The inductor and capacitor are in parallel, therefore $v_L(0^+) = 0\text{V}$.

$$\boxed{\left.\frac{di_L}{dt}\right|_{t=0^+} = \frac{v_L(0^+)}{L} = 0\text{A/s}}$$

From the $t = 0^+$ solution, $i_C(0^+) = 0.28\text{mA}$. $\left.\frac{dv_c}{dt}\right|_{t=0^+} = \frac{i_c(0^+)}{C} = 280\text{V/s}$.

From KVL, $80\text{V}=45\text{k}\Omega i_R + v_C$ Taking the derivative of both sides and evaluating at $t = 0^+$: $\left.\frac{d80}{dt}\right|_{t=0^+} = 45\text{k}\Omega \left.\frac{di_R}{dt}\right|_{t=0^+} + \left.\frac{dv_C}{dt}\right|_{t=0^+}$.

$$\boxed{\frac{di_R}{dt} = -\frac{1}{45\text{k}\Omega} \times \left.\frac{dv_C}{dt}\right|_{t=0^+} = -\frac{280\text{V/s}}{45\text{k}\Omega} = -6.18\text{A/s}}$$

Finally (!!) use KCL: $i_R = i_C + i_L$ and differentiate both sides.

$$\frac{di_R}{dt} = \frac{di_C}{dt} + \frac{di_L}{dt}, \text{ or } \frac{di_R}{dt} - \frac{di_L}{dt} = \boxed{\frac{di_C}{dt} = -6.18\text{A/s} - 0\text{A/s} = -6.18\text{A/s}}$$

8. Second order initial conditions

The 3A source is OFF (0 A) for $t < 0$ and suddenly turns on at $t = 0$. (This is essentially $3u(t)$ V).

Obtain the initial and final conditions necessary to solve for $v(t)$ and $i(t)$ for $t > 0$.

OK, we need a bunch of initial conditions. For $t < 0$, the 3A source is off, an open circuit.

The capacitor is open, so no current flows. KVL: $-20\text{V} + v_C(0^-) = 0$, $\boxed{v_C(0^-) = 20\text{V}}$.

The inductor is a wire, but no current flows due to the capacitor, $\boxed{i_L(0^-) = 0\text{A}.}$

For $t = \infty$, the capacitor is open, no current flows. But the 3A||5Ω is 15V in series with 5Ω. KVL: $-20\text{V} - 15\text{V} + v_C = 0$, $\boxed{v_C(\infty) = 35\text{V}.}$

The capacitor is open, so no current flows, so $\boxed{i_L(\infty) = 0\text{A}}$.

Now the derivatives.

At $t = 0^+$, $-20\text{V} - 15\text{V} + i(0^+)(2\Omega + 1\Omega + 5\Omega) + v_L(0^+) + v_C(0^+) = 0$

$35\text{V} = 0\text{A} \times (8\Omega) + v_L(0^+) + 20\text{V}$

$$15\text{V} = v_L(0^+) \rightarrow \boxed{\left.\frac{di_L}{dt}\right|_0 = \frac{15\text{V}}{5\text{H}} = 3\text{A/s}}$$

After the source transformation, all elements are in series, so $i_C(0^+) = i_L(0^+) = 0\text{A}$.

$$\text{Thus } \boxed{\left.\frac{dv_C}{dt}\right|_0 = \frac{i_C(0^+)}{C} = 0 \text{ V/s.}}$$

Now to the solution. After the source transformation, everything is in series, so this is series LRC circuit. $\alpha = \frac{R}{2L} = \frac{2+5+1\Omega}{2\times 5H\text{H}} = 0.8/\text{s}$. $\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{5\text{H}\times 0.2\text{F}}} = 1 \text{ rad/s}$.

$\alpha < \omega_0$ so this is underdamped. $\omega_d = \sqrt{1^2 - 0.8^2} = 0.6$ rad/s.

The general solution is $v(t) = v(\infty) + e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t)$.

$v_C(0) = v_C(\infty) + B_1$; $20\text{V} = 35\text{V} + B_1$, so $B_1 = -15\text{V}$.

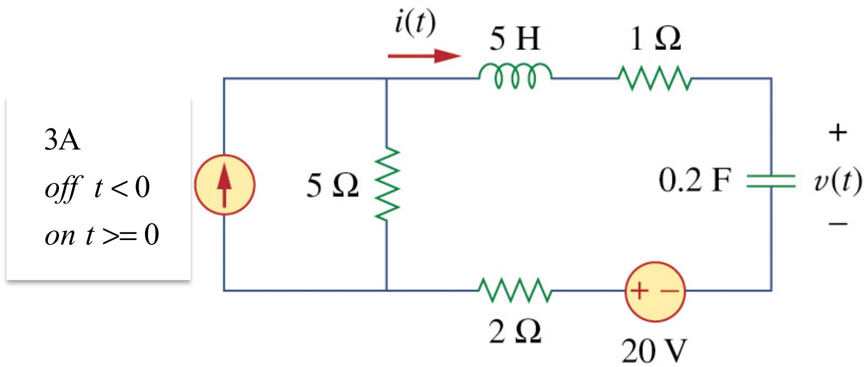
$$\left.\frac{dv_C}{dt}\right|_{t=0^+} = -\alpha B_1 + \omega_d B_2;\ 0 = -0.8/\text{s} \times (-15\text{V}) + 0.6/\text{s} \times B_2$$

$$\frac{-12\text{V/s}}{0.6/\text{s}} = -20\text{V/s} = B_2.$$

$$\boxed{v(t) = 35 - e^{-0.8t}(15\cos 0.6t + 20\sin 0.6t)\text{V}}$$

Just checking, $t = 0, 35 - 15 = 20$

$$t = \infty, 35 = 35;\ \left.\frac{dv_C}{dt}\right|_{t=0^+} = 0 = -0.8(-15) + 0.6(-20) = 12 - 12 = 0.$$



9. Second order solution

Find $v(t), t > 0$ in the previous circuit. *Hint: Is it serial LRC? How is it damped?*

Now to the solution. After the source transformation, everything is in series, so this is series LRC circuit. $\alpha = \frac{R}{2L} = \frac{2+5+1\Omega}{2\times 5H\text{H}} = 0.8/\text{s}$. $\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{5\text{H}\times 0.2\text{F}}} = 1$ rad/s.

$\alpha < \omega_0$ so this is underdamped. $\omega_d = \sqrt{1^2 - 0.8^2} = 0.6$ rad/s.

The general solution is $v(t) = v(\infty) + e^{-\alpha t}\left(B_1 \cos\omega_d t + B_2 \sin\omega_d t\right)$.

$v_C(0) = v_C(\infty) + B_1$; $20\text{V} = 35\text{V} + B_1$, so $B_1 = -15\text{V}$.

$$\left.\frac{dv_C}{dt}\right|_{t=0^+} = -\alpha B_1 + \omega_d B_2;\ 0 = -0.8/\text{s}\times(-15\text{V}) + 0.6/\text{s}\times B_2$$

$$\frac{-12\text{V/s}}{0.6/\text{s}} = -20\text{V/s} = B_2.$$

$$\boxed{v(t) = 35 - e^{-0.8t}\left(15\cos 0.6t + 20\sin 0.6t\right)\text{V}}$$

Just checking, $t = 0, 35 - 15 = 20$

$$t = \infty, 35 = 35;\ \left.\frac{dv_C}{dt}\right|_{t=0^+} = 0 = -0.8\left(-15\right) + 0.6(-20) = 12 - 12 = 0.$$